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Bounds for Certain Freud-Type Orthogonal Polynomials

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Let $w_Q(x) = \exp(-Q(x))$ be a weight function and $\{p_n\}$ the system of polynomials orthonormal with respect to w_Q^2 on $\mathbf{R}$. We show that if Q satisfies certain technical conditions, then

$$|w_Q(x)\, p_n(x)| \leqslant c_1 q_n^{-1/2}, \qquad \text{for} \quad |x| \leqslant c_2 q_n,\ n = 1, 2, 3, ...,$$

where c_1, c_2 are constants depending upon Q alone and $q_n Q'(q_n) = n$, $n = 1, 2, ...$ The weights considered include $\exp(-|x|^\alpha)$ when $\alpha \geqslant 4$. The proof involves the use of certain "infinite–finite range inequalities" to estimate the coefficients in a differential equation satisfied by p_n. These estimates, in turn, enables us to use a concavity argument. © 1990 Academic Press, Inc.

1. INTRODUCTION

One of the classical inequalities for the orthonormal Hermite polynomials h_n (orthonormal on $\mathbf{R}$ with respect to the weight function $\exp(-x^2)$) is the following

THEOREM 1.1 [24, Theorem 8.22.9]. *Let $\varepsilon > 0$. Then there exists a positive constant $c(\varepsilon)$ such that for all real x with $|x| \leqslant (1-\varepsilon)\sqrt{2n}$,*

$$|\exp(-x^2/2)\, h_n(x)| \leqslant c(\varepsilon)\, n^{-1/4}. \tag{1.1}$$

In recent years, there has been considerable interest in obtaining generalizations and refinements of this theorem for Freud polynomials, i.e., polynomials orthonormal on $\mathbf{R}$ with respect to a weight function of the form $\exp(-2Q(x))$ [22, 23, 13, 14, 3, 4, 1]. A typical result is the following

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THEOREM 1.2 [4, 13]. *Let m be an even positive integer and $\{p_n\}$ the system of polynomials orthonormal on* **R** *with respect to* $\exp(-x^m)$. *Then,*

$$|\exp(-x^m/2)\, p_n(x)((2\beta n^{1/m})^2 - x^2)^{1/4}| \leqslant c$$
$$\text{for} \quad |x| \leqslant 2\beta n^{1/m},\ n = 1, 2, \ldots, \tag{1.2}$$

where c is a constant depending upon m alone and

$$\beta := \left\{\frac{\sqrt{\pi}\, \Gamma(m/2)}{\Gamma(m+1/2)}\right\}^{1/m}. \tag{1.3}$$

All of the results known to the author in this direction concern the case when Q is a polynomial. (See, however, the announcement in [12].) An essential ingredient of the proofs is detailed information about the asymptotic behavior of the recurrence coefficients for the orthogonal polynomials.

In this paper, we obtain an analogue of Theorem 1.2 for polynomials orthonormal on **R** with respect to a weight function of the form $\exp(-2Q(x))$ where Q is a "general" function. The main idea is essentially to use a differential equation satisfied by these polynomials and a concavity argument as in [4]. However, practically no information is known about the recurrence coefficients for the "general" weight functions. We shall rely upon our results with Saff in [20] to obtain some relatively crude estimates on these coefficients and other quantities in the differential equation. These estimates will then help us to deduce an analogue of Theorem 1.2. Of course, our result is not as sharp as Theorem 1.2 but is sufficient in many applications [9–11, 18, 23].

After the first draft of this manuscript was submitted, we learned from Doron Lubinsky that he has, in fact, obtained the Plancherel–Rotach-type asymptotics for polynomials orthogonal on **R** with respect to a weight function belonging to a fairly general class of weights. His results imply our results as a special case, and are valid, in particular, for the weights $\exp(-|x|^\alpha)$ when $\alpha > 3$. To the best of our knowledge, our approach in general, and, in particular, the simplification of the differential equation as well as the estimation of the various quantities appearing in the equation remain as the novel features of this paper. The simplification, in fact, has been used in the study of Hermite interpolation based at the zeros of Freud polynomials [21]. Using the ideas of [21], it seems possible that our conditions on the weight function can be relaxed somewhat so as to include the weights $\exp(-|x|^\alpha)$ when $\alpha > 3$. In the light of Lubinsky's results, we choose not to do so in this paper.

In the next section, we state the precise conditions on the weight function and discuss the main results. These results will be proved in Section 3.

I thank Paul Nevai, for his kind encouragement in this work, including making [1, 4] available to me as well as Doron Lubinsky for his several comments.

2. Main Results

We shall consider weight functions of the form w_Q^2 where $w_Q(x) := \exp(-Q(x))$. Throughout this paper, we shall adopt the following convention. The symbols c, c_1, ... will denote constants depending upon Q alone. Their values may be different at different occurrences, even within the same formula. $A \sim B$ will mean $c_1 A \leqslant B \leqslant c_2 A$. The class of all polynomials of degree not exceeding n will be denoted by Π_n.

We shall assume the following conditions on w_Q.

(W1) Q is an even, convex function in $C^4(\mathbf{R})$ and $tQ'(t) \to \infty$ as $|t| \to \infty$.

(W2) Let q_n be the least positive solution of the equation

$$q_n Q'(q_n) = n. \tag{2.1}$$

Then for every $c > 0$ and $c_1 < |t| < |x| \leqslant cq_n$,

$$\left|\frac{Q'(t)}{t^3}\right| \leqslant c_2 \left|\frac{Q'(x)}{x^3}\right| \leqslant c_3 n q_n^{-4}, \tag{2.2a}$$

$$\left|\frac{t^3 Q^{(4)}(t)}{Q'(t)}\right| \leqslant c_2. \tag{2.2b}$$

(W3) *For the numbers* q_n *defined in* (2.1), $q_{2n} \sim q_n$.

We do not claim that these conditions are independent. They are all satisfied when $Q(x) = |x|^\alpha$, $\alpha \geqslant 4$, which are some of the prototypical Freud weights. Another example is the weight $|x|^\beta \exp(-|x|^\alpha)$, $\alpha \geqslant 4$, $\beta > 0$. We note the following simple consequences of our conditions which will be needed later.

$$Q(|x|) \leqslant Q(y) \qquad \text{if} \quad |x| \leqslant y \tag{2.3}$$

$$\left|\frac{xQ''(x)}{Q'(x)}\right| \leqslant c \qquad \text{if} \quad |x| \geqslant c_1 \tag{2.4}$$

$$\left|\frac{x^2 Q'''(x)}{Q'(x)}\right| \leqslant c \qquad \text{if} \quad |x| \geqslant c_1. \tag{2.5}$$

A consequence of our results with Saff in [19, 20] is the following

THEOREM 2.1. *Let $0<p, r<\infty$. There exists a sequence $\{a_n(Q)\}$ and positive constants c_1, c_2 depending upon p, r, Q alone such that for every $n=1, 2, ...$ and $P \in \Pi_n$,*

$$|w_Q(x)\,P(x)|^r \leqslant c_1 \exp(-c_2 n) \int_{|t| \leqslant a_n(Q)} |w_Q(t)\,P(t)|^r\,dt, \qquad |x| \geqslant a_n(Q), \tag{2.6}$$

$$\left\{\int_{|x| \geqslant a_n(Q)} |w_Q(x)\,P(x)|^p\,dx\right\}^{r/p} \leqslant c_1 \exp(-c_2 n) \int_{|t| \leqslant a_n(Q)} |w_Q(t)\,P(t)|^r\,dt, \qquad |x| \geqslant a_n(Q). \tag{2.7}$$

We caution the reader that the notation here is different from that in [19, 20]. We can, in fact, take [19]

$$a_n(Q) \sim q_n. \tag{2.8a}$$

In particular, for any $p, r>0$,

$$a_n(Q) \sim a_{pn}(rQ). \tag{2.8b}$$

We shall denote $a_n(Q/2)$ by a_n.

Let $\{p_n\}$ denote the system of polynomials orthonormal on $\mathbf{R}$ with respect to w_Q^2.

$$p_n(x) = \gamma_n x^n + \cdots = \gamma_n \prod_{k=1}^{n} (x - x_{k,n}) \tag{2.9a}$$

$$\gamma_n > 0, \qquad x_{n,n} < x_{n-1,n} < \cdots < x_{1,n} \tag{2.9b}$$

$$\int_{-\infty}^{\infty} p_n(t)\,p_m(t)\,w_Q^2(t)\,dt = \delta_{nm}. \tag{2.9c}$$

Our main theorem is the following.

THEOREM 2.2. *If Q satisfies the conditions (W1), (W2), (W3), then there exist constants $c_1, c_2>0$ such that*

$$|w_Q(x)\,p_n(x)| \leqslant c_1 a_n^{-1/2} \qquad \textit{if} \quad |x| \leqslant c_2 a_n. \tag{2.10}$$

We note the following consequences of our theorem which might be of some interest in applications.

COROLLARY 2.3. *There exists a constant c with the property that for any* $\varepsilon > 0$,

$$\int_{|x| \geq c\varepsilon a_n} p_n^2(x)\, w_Q^2(x)\, dx > 1 - \varepsilon, \qquad n = 1, 2, \ldots. \tag{2.11}$$

COROLLARY 2.4. (cf. [3. 14]). *For* $n = 1, 2, \ldots$,

$$c_1 a_n^{-1/2} \leq |w_Q(x_{k,n})\, p_{n-1}(x_{k,n})| \leq c_2 a_n^{-1/2} \qquad \text{if} \quad |x_{k,n}| \leq c_3 a_n. \tag{2.12}$$

It seems probable that a more precise version of Theorem 2.2, similar to Theorem 1.2, is true. The techniques known to the author for proving such inequalities, however, require that the sequence $\{a_n^{-1}\gamma_{n-1}/\gamma_n\}$ converges at a "good" rate as $n \to \infty$. While the convergence itself is known for a large class of weight functions [15], the rate is known only in the case when Q is a polynomial of even degree and positive leading coefficient [2, 16, 17].

3. PROOFS

The first step in our proof of Theorem 2.2 is to obtain a differential equation satisfied by p_n. This equation does not really require the conditions $(W1)$, $(W2)$, and $(W3)$ on Q. In each of the results stated below while obtaining this equation, we assume only that w_Q is a weight function and Q has sufficiently many continuous derivatives to make the various quantities involved well defined.

We begin by recalling certain standard properties of orthogonal polynomials.

PROPOSITION 3.1 [5]. (a)

$$x p_{n-1}(x) = \rho_n p_n(x) + \beta_n p_{n-1}(x) + \rho_{n-1} p_{n-2}(x), \qquad x \in \mathbf{R},\ n = 2, 3, \ldots, \tag{3.1a}$$

where

$$\rho_n := \gamma_{n-1}/\gamma_n, \qquad \beta_n \in \mathbf{R}. \tag{3.1b}$$

(b) *For every polynomial* $P \in \Pi_{n-1}$, $x \in \mathbf{R}$,

$$P(x) = \int P(t)\, K_n(x, t)\, w_Q^2(t)\, dt, \tag{3.2a}$$

where

$$K_n(x, t) := \sum_{k=0}^{n-1} p_k(x)\, p_k(t) = \rho_n \frac{p_n(x)\, p_{n-1}(t) - p_n(t)\, p_{n-1}(x)}{x - t}. \tag{3.2b}$$

(c) *Let*

$$\lambda_n(x) := \{K_n(x, x)\}^{-1}. \tag{3.3}$$

Then

$$\lambda_n^{-1}(x) = \rho_n\{p_n'(x)\, p_{n-1}(x) - p_n(x)\, p_{n-1}'(x)\}. \tag{3.4}$$

In particular,

$$\lambda_{kn}^{-1} := \lambda_n^{-1}(x_{k,n}) = \rho_n\, p_n'(x_{k,n})\, p_{n-1}(x_{k,n}). \tag{3.5}$$

(d) *For every* $P \in \Pi_{2n-1}$,

$$\int P(t)\, w_Q^2(t)\, dt = \sum_{k=1}^{n} \lambda_{kn} P(x_{k,n}). \tag{3.6}$$

(e)

$$\int p_n'^2(t)\, w_Q^2(t)\, dt \geqslant n^2/\rho_n^2, \qquad n = 1, 2, ... \tag{3.7}$$

Next, we obtain an expression for p_n' in terms of p_n and p_{n-1} (cf. [4]).

THEOREM 3.2. *For* $n = 1, 2, ..., x \in \mathbf{R}$,

$$p_n'(x) = A_n(x)\, p_{n-1}(x) - B_n(x)\, p_n(x), \tag{3.8a}$$

where, with

$$\bar{Q}(x, t) := \frac{Q'(t) - Q'(x)}{t - x}, \tag{3.8b}$$

we have

$$A_n(x) := 2\rho_n \int p_n^2(t)\, w_Q^2(t)\, \bar{Q}(x, t)\, dt \tag{3.8c}$$

$$B_n(x) := 2\rho_n \int p_n(t)\, p_{n-1}(t)\, w_Q^2(t)\, \bar{Q}(x, t)\, dt. \tag{3.8d}$$

Proof of Theorem 3.2. In view of Proposition 3.1(b),

$$p_n'(x) = \int p_n'(t)\, K_n(x, t)\, w_Q^2(t)\, dt. \tag{3.9}$$

If we integrate by parts and observe that p_n is orthogonal to $(\partial/\partial t)\, K_n(x, t)$, which is a polynomial of degree at most $n-2$ in t, then we get

$$p_n'(x) = 2 \int p_n(t)\, K_n(x, t)\, Q'(t)\, w_Q^2(t)\, dt. \tag{3.10}$$

Since p_n is orthogonal to $K_n(x, t)$, we see that for any constant K,

$$p_n'(x) = 2 \int p_n(t)\, K_n(x, t)(Q'(t) - K)\, w_Q^2(t)\, dt. \tag{3.11}$$

We choose $K = Q'(x)$ and use (3.2b) to arrive at (3.8). ∎

We note one corollary of (3.11) which will be used in the sequel.

COROLLARY 3.3. *For any $K \in \mathbf{R}$ and $n = 1, 2, ...,$*

$$\int p_n^2(t)(Q'(t) - K)^2\, w_Q^2(t)\, dt \geqslant \frac{n^2}{4\rho_n^2}. \tag{3.12}$$

Proof. Equation (3.11) shows that $p_n'(x)/2$ is the nth partial sum of the orthonormal expansion of the function $p_n(x)(Q'(t) - K)$. We get (3.12) by using Bessel's inequality and (3.7). ∎

Next, we use some ideas originating from Shohat and also from Nevai, and perform some elementary computations based on (3.1a) and (3.8a) to obtain, as in [4, 1],

THEOREM 3.4. *For $n = 2, 3, ...,$ and $x \in \mathbf{R}$,*

$$p_n''(x) + M_n(x)\, p_n'(x) + N_n(x)\, p_n(x) = 0, \tag{3.13a}$$

where

$$M_n(x) := B_n(x) + B_{n-1}(x) - \frac{x - \beta_n}{\rho_{n-1}} A_{n-1}(x) - \frac{A_n'(x)}{A_n(x)} \tag{3.13b}$$

$$N_n(x) := \frac{A_n(x)\, A_{n-1}(x)\, \rho_n}{\rho_{n-1}} - \frac{A_{n-1}(x)\, B_n(x)(x - \beta_n)}{\rho_{n-1}}$$
$$+ B_n(x)\, B_{n-1}(x) + B_n'(x) - \frac{A_n'(x)}{A_n(x)} B_n(x). \tag{3.13c}$$

We pause to make the following observation which will simplify the expressions for M_n and N_n.

PROPOSITION 3.5. *For* $n=2, 3, ..., x \in \mathbf{R}$,

$$B_n(x)+B_{n-1}(x)-\frac{x-\beta_n}{\rho_{n-1}}A_{n-1}(x)=-2Q'(x). \tag{3.14}$$

Proof. Using (3.8) and (3.1a), we get

$$\begin{aligned} &B_n(x)+B_{n-1}(x)\\ &\quad=2\int p_{n-1}(t)\{\rho_n p_n(t)+\rho_{n-1}p_{n-2}(t)\}\,\bar{Q}(x,t)\,w_Q^2(t)\,dt\\ &\quad=2\int p_{n-1}^2(t)(t-\beta_n)\,\bar{Q}(x,t)\,w_Q^2(t)\,dt\\ &\quad=2\int p_{n-1}^2(t)\{Q'(t)-Q'(x)\}\,w_Q^2(t)\,dt+\frac{x-\beta_n}{\rho_{n-1}}A_{n-1}(x)\\ &\quad=-2Q'(x)+\frac{x-\beta_n}{\rho_{n-1}}A_{n-1}(x)+2\int p_{n-1}^2(t)\,Q'(t)\,w_Q^2(t)\,dt. \end{aligned} \tag{3.15}$$

Using integration by parts, we see that

$$2\int p_{n-1}^2(t)\,Q'(t)\,w_Q^2(t)\,dt=\int p_{n-1}(t)\,p_{n-1}'(t)\,w_Q^2(t)\,dt=0.$$

Thus, (3.15) gives (3.14). ∎

Following [4, 1], we next use a standard technique in the theory of differential equations to convert Eq. (3.13a) into an equation without the "middle term." We note first that since Q is convex, $A_n(x)>0$ for every $x \in \mathbf{R}$.

THEOREM 3.6. *For* $n=2, 3, ..., x \in \mathbf{R}$, *let*

$$z_n(x):=p_n(x)\,w_Q^2(x)\,A_n^{-1/2}(x). \tag{3.16}$$

Then

$$z_n''(x)+\Phi_n(x)\,z_n(x)=0, \tag{3.17a}$$

where

$$\Phi_n:=N_n-\frac{1}{4}M_n^2-\frac{1}{2}M_n'. \tag{3.17b}$$

The proof of Theorem 3.6 is an elementary computation and hence is omitted. The only remark we wish to make is that there is no "correction term" in the expression for z_n, as a result of (3.14).

Next, we proceed to show that, in fact, $\Phi_n(x) \sim (n/a_n)^2$ if $|x| \leqslant ca_n$. This is the part where we require all the assumptions (W1), (W2), and (W3) on the weight function. These are therefore assumed in the sequel.

First, we obtain some estimates on ρ_n and A_n. We do not claim that the estimates on ρ_n are new [6–8], but since they are not difficult, we include the proof.

PROPOSITION 3.7. *Let Q satisfy the conditions* (W1), (W2), *and* (W3) *defined in the beginning of Section* 2. *We have*

$$c_1 a_n \leqslant \rho_n \leqslant c_2 a_n. \tag{3.18}$$

For every $L > 1$,

$$c_1(L) \frac{n}{a_n} \leqslant A_n(x) \leqslant c_2(L) \frac{n}{a_n}, \qquad \textit{if} \quad |x| \leqslant La_n. \tag{3.19}$$

A critical aspect of the proof is to prove certain "infinite–finite range inequalities" for the integrals which arise in connection with ρ_n and $A_n(x)$.

LEMMA 3.8. *Let Q satisfy the conditions* (W1), (W2), *and* (W3) *defined in the beginning of Section* 2.

(a)

$$\int_{|t| \geqslant y} Q'^2(t)\, w_Q(t)\, dt \leqslant cQ'(y)\, w_Q(y) \qquad \textit{if} \quad y > c_1. \tag{3.20}$$

(b)

$$\int_{|t| \geqslant a_n} p_n^2(t)\, Q'^2(t)\, w_Q^2(t)\, dt \leqslant c_1 \exp(-c_2 n). \tag{3.21}$$

(c) *If $L > 1$, $|x| \leqslant La_n$, and $\varepsilon > 0$, then*

$$\left| (2\rho_n)^{-1} A_n(x) - \int_{|t| \leqslant (L+\varepsilon) a_n} p_n^2(t)\, \bar{Q}(x, t)\, w_Q^2(t)\, dt \right| \leqslant \frac{c_1}{\varepsilon} \exp(-c_2 n), \tag{3.22}$$

where c_1, c_2 may depend upon L.

Proof of Lemma 3.8. (a) Using (2.4) and (W1), we observe that $Q''(t)/Q'^2(t) \to 0$ as $|t| \to \infty$. Consequently, if $|t| > c$, then

$$w_Q''(t) = \{Q'^2(t) - Q''(t)\}\, w_Q(t) \sim Q'^2(t)\, w_Q(t). \tag{3.23}$$

The estimate (3.20) now follows upon integrating (3.23).

(b) Using (2.6) with $r=2$ and $Q/2$ in place of Q, we get

$$\begin{aligned} w_Q(t)\,p_n^2(t) &\leqslant c_1 \exp(-c_2 n) \int_{|u| \leqslant a_n} w_Q(u)\,p_n^2(u)\,du \\ &\leqslant c_1 \exp(-c_2 n)\,w_Q^{-1}(a_n) \qquad \text{if} \quad |t| \geqslant a_n. \end{aligned} \tag{3.24}$$

Hence, if n is large enough, (3.20) yields

$$\begin{aligned} &\int_{|t| \geqslant a_n} p_n^2(t)\,Q'^2(t)\,w_Q^2(t)\,dt \\ &\quad \leqslant c_1 \exp(-c_2 n)\,w_Q^{-1}(a_n) \int_{|t| \geqslant a_n} Q'^2(t)\,w_Q(t)\,dt \\ &\quad \leqslant c_1 \exp(-c_2 n)\,Q'(a_n) \\ &\quad \leqslant c_1 \exp(-c_3 n). \end{aligned}$$

(c) When $|x| \leqslant La_n$ and $|t| \geqslant (L+\varepsilon)\,a_n$ then $|t-x| \geqslant \varepsilon a_n$, while

$$|Q'(t)-Q'(x)| \leqslant c\,|Q'(t)| \leqslant cQ'^2(t).$$

Hence, (3.21) yields that

$$\int_{|t| \geqslant (L+\varepsilon)\,a_n} p_n^2(t)\,\bar{Q}(x,t)\,w_Q^2(t)\,dt \leqslant \frac{c}{\varepsilon a_n} \exp(-c_2 n) \leqslant \frac{c}{\varepsilon} \exp(-c_2 n).$$

The estimate (3.22) now follows from (3.8c). ∎

Proof of Proposition 3.7. First, we observe that an application of the Schwarz inequality gives, for every $A>0$,

$$\int_{|t| \leqslant A} |tp_n(t)\,p_{n-1}(t)\,w_Q^2(t)|\,dt \leqslant A. \tag{3.25}$$

Using Theorem 2.1, (2.8), and (3.25) with $A = ca_n$ for an appropriate choice of c, we get

$$\int_{|t| \geqslant ca_n} |tp_n(t)\,p_{n-1}(t)\,w_Q^2(t)|\,dt \leqslant c_1 \exp(-c_2 n). \tag{3.26}$$

In view of (3.25), (3.26), we have

$$\begin{aligned} \rho_n &\leqslant \int |tp_n(t)\,p_{n-1}(t)\,w_Q^2(t)|\,dt \\ &= \left(\int_{|t| \leqslant ca_n} + \int_{|t| \geqslant ca_n}\right) |tp_n(t)\,p_{n-1}(t)\,w_Q^2(t)|\,dt \\ &\leqslant ca_n + c_1 \exp(-c_2 n) \leqslant ca_n. \end{aligned} \tag{3.27}$$

Next, lt $L>1$. In view of the mean value theorem, (2.4), and (2.2), when $|x|\leqslant La_n$,

$$\int_{|t|\leqslant 2La_n} p_n^2(t)\,\bar{Q}(x,t)\,w_Q^2(t)\,dt$$

$$\leqslant c\frac{n}{a_n^2}\int_{|t|\leqslant 2La_n} p_n^2(t)\,w_Q^2(t)\,dt\leqslant c\frac{n}{a_n^2}. \tag{3.28}$$

Therefore, (3.27) and (3.22) (with $\varepsilon=L$) imply that, for every $L>1$,

$$|A_n(x)|\leqslant c(L)\frac{n}{a_n} \qquad \text{if} \quad |x|\leqslant La_n. \tag{3.29}$$

In view of (2.8), Theorem 2.1, Theorem 3.2, we see that for sufficiently large L,

$$\int p_n'^2(t)\,w_Q^2(t)\,dt\leqslant 2\int_{|t|\leqslant La_n} |p_n'(t)\,w_Q(t)|^2\,dt$$

$$\leqslant c\left\{\int_{|t|\leqslant La_n} A_n^2(t)\,p_{n-1}^2(t)\,w_Q^2(t)\,dt\right.$$

$$\left.+\int_{|t|\leqslant La_n} B_n^2(t)\,p_n^2(t)\,w_Q^2(t)\,dt\right\}. \tag{3.30}$$

Now, (3.29) implies that

$$\int_{|t|\leqslant La_n} A_n^2(t)\,p_{n-1}^2(t)\,w_Q^2(t)\,dt\leqslant c\left(\frac{n}{a_n}\right)^2. \tag{3.31}$$

Using the Schwarz inequality in (3.8d), and the estimates (3.27), (3.28), it is readily seen that

$$|B_n(t)|^2\leqslant c\left(\frac{n}{a_n}\right)^2, \qquad |t|\leqslant La_n, \qquad \text{for every} \quad L>1. \tag{3.32}$$

Substituting from (3.32) and (3.31) into (3.30), we get

$$\int p_n'^2(t)\,w_Q^2(t)\,dt\leqslant c\left(\frac{n}{a_n}\right)^2.$$

In view of (3.7), this yields $\rho_n\geqslant ca_n$.

Finally, we prove the first inequality in (3.19). Let $L>1$. If $|x|\leqslant La_n$, then it is not difficult to deduce, using Lemma 3.8(b) and (2.2) that

$$\int_{|t|\geqslant 2La_n} p_n^2(t)\,w_Q^2(t)\{Q'(t)-Q'(x)\}^2\,dt\leqslant c_1\exp(-c_2 n). \tag{3.33}$$

Consequently, (3.18) and (3.12) yield that

$$\int_{|t| \leqslant 2La_n} p_n^2(t)\, w_Q^2(t)\{Q'(t)-Q'(x)\}^2\, dt \geqslant c\,\frac{n^2}{a_n^2}. \tag{3.34}$$

Next, if $|t| \leqslant 2La_n$, then

$$|(t-x)(Q'(t)-Q'(x))| \leqslant ca_n Q'(a_n) \leqslant cn.$$

So, since Q is convex, (3.34) implies that

$$\begin{aligned} &\int_{|t| \leqslant 2La_n} p_n^2(t)\, \bar{Q}(x,t)\, w_Q^2(t)\, dt \\ &\geqslant \frac{c}{n} \int_{|t| \leqslant 2La_n} p_n^2(t)\, w_Q^2(t)\{Q'(t)-Q'(x)\}^2\, dt \\ &\geqslant c\,\frac{n}{a_n^2}. \end{aligned} \tag{3.35}$$

The first inequality in (3.19) now follows from (3.22) and (3.18). ∎

We also need estimates on A_n', B_n', A_n''. These are obtained exactly in the same way as the upper estimates on A_n, using the mean value theorem, (2.2), (2.4), (2.5), and Lemma 3.8. We omit the proofs, but note the result below.

PROPOSITION 3.9. *Let Q satisfy the conditions* (W1), (W2), *and* (W3) *defined in the beginning of Section* 2. *Let $L > 1$. Then, for $|x| \leqslant La_n$,*

$$|A_n^{(r)}(x)| \leqslant c(L)\, na_n^{-r-1}, \qquad r = 0, 1, 2. \tag{3.36}$$

$$|B_n^{(r)}(x)| \leqslant c(L)\, na_n^{-r-1}, \qquad r = 0, 1, 2. \tag{3.37}$$

We would like to observe that in proving Propositions 3.7 and 3.9, we did not really need the fact that Q is even. In order to estimate Φ_n, we will, however, need a more refined estimate on B_n. At this time, we are able to do so only when Q is even.

PROPOSITION 3.10. *Let Q satisfy the conditions* (W1), (W2), *and* (W3) *defined in the beginning of Section* 2. *For any $L > 1$ and $|x| \leqslant La_n$,*

$$|B_n(x)| \leqslant c(L)\,\frac{n}{a_n}\,\frac{|x|}{a_n}. \tag{3.38}$$

Proof. Since Q is even, $p_n(t)\,p_{n-1}(t)$ is an odd function of t. Consequently, (3.8d) shows that $B_n(0)=0$. Estimate (3.38) now follows from the mean value theorem and (3.37). ∎

We are now in a position to estimate Φ_n.

THEOREM 3.11. *Let Q satisfy the conditions* (W1), (W2), *and* (W3) *defined in the beginning of Section* 2. *There exists a positive constant α, depending upon Q alone, such that for $|x|\leqslant\alpha a_n$,*

$$\Phi_n(x)\sim\left(\frac{n}{a_n}\right)^2. \tag{3.39}$$

Proof. Since Q is even, the parameter β_n in (3.1a) is zero. Consequently, (3.13c) becomes

$$\begin{aligned}N_n(x):=&\frac{A_n(x)\,A_{n-1}(x)\,\rho_n}{\rho_{n-1}}-\frac{A_{n-1}(x)\,B_n(x)\,x}{\rho_{n-1}}\\&+B_n(x)\,B_{n-1}(x)+B_n'(x)-\frac{A_n'(x)}{A_n(x)}B_n(x).\end{aligned} \tag{3.40}$$

Let $|x|\leqslant\alpha a_n$, where $0<\alpha<1$ will be chosen later. In view of Propositions 3.7, 3.9,

$$|N_n(x)|\leqslant c\left(\frac{n}{a_n}\right)^2. \tag{3.41}$$

Propositions 3.7, 3.9, 3.10 also yield that

$$\frac{A_nA_{n-1}a_n\rho_n}{\rho_{n-1}}\geqslant m_1\left(\frac{n}{a_n}\right)^2 \tag{3.42a}$$

$$\left|N_n-\frac{A_{n-1}A_n\rho_n}{\rho_{n-1}}\right|\leqslant m_2\alpha^2\left(\frac{n}{a_n}\right)^2, \tag{3.42b}$$

where m_1 and m_2 are constants depending only on Q. Since $\alpha\leqslant 1$, they do not depend upon α, as it might appear. Next, Propositions 3.5, 3.9, and estimates (2.2) yield that

$$M_n^2(x)=\left(2Q'(x)+\frac{A_n'(x)}{A_n(x)}\right)^2\leqslant m_3\alpha^2\left(\frac{n}{a_n}\right)^2. \tag{3.43}$$

Propositions 3.5, 3.9, and estimates (2.2) also show that

$$|M_n'(x)|\leqslant\frac{c}{n}\left(\frac{n}{a_n}\right)^2. \tag{3.44}$$

Substituting from (3.44), (3.43), and (3.41) into (3.17), we see that

$$|\Phi_n(x)| \leqslant c\left(\frac{n}{a_n}\right)^2. \tag{3.45}$$

Substituting from (3.44), (3.43), and (3.42) into (3.17), we get

$$\Phi_n(x) \geqslant \left(m_1 - m_2\alpha^2 - \frac{m_3}{4}\alpha^2\right)\left(\frac{n}{a_n}\right)^2 - \frac{c}{n}\left(\frac{n}{a_n}\right)^2. \tag{3.46}$$

We now choose

$$\alpha := \sqrt{m_1/(4m_2+m_3)} \tag{3.47}$$

to get, for $|x| \leqslant \alpha a_n$,

$$\Phi_n(x) \geqslant \left(\frac{3m_1}{4} - \frac{c}{n}\right)\left(\frac{n}{a_n}\right)^2 \geqslant c\left(\frac{n}{a_n}\right)^2. \quad \blacksquare \tag{3.48}$$

The remainder of our paper uses the same argument as in [4, 1]. First, if we compare the differential equation

$$Y'' + c\left(\frac{n}{a_n}\right)^2 Y = 0 \tag{3.49}$$

for judiciously chosen constants c with (3.17), then Sturm's comparison theorem [24] with Theorem 3.11 yields the following

COROLLARY 3.12 (cf. [8]). *Let Q satisfy the conditions* (W1), (W2), *and* (W3) *defined in the beginning of Section* 2. *For the consecutive zeros* $x_{k,n}$, $x_{k+1,n}$ *of* p_n, *which lie in the interval* $[-\alpha a_n, \alpha a_n]$, *we have*

$$x_{k,n} - x_{k+1,n} \sim a_n/n. \tag{3.50}$$

Next, we need a technical estimate.

LEMMA 3.13. *Let Q satisfy the conditions* (W1), (W2), *and* (W3) *defined in the beginning of Section* 2. *With the numbers* λ_{kn} *defined in* (3.5), *we have*

$$\lambda_{kn}\, p_n'^2(x_{k,n}) = A_n(x_{k,n})\, \rho_n^{-1} \leqslant cna_n^{-2}. \tag{3.51}$$

Proof. The equality in (3.51) follows from (3.5) and (3.8a); the inequality follows from Proposition 3.7. ∎

Proof of Theorem 2.2. Let x be fixed such that $|x| \leqslant \alpha a_n/2$, $x \in [x_{l+1,n}, x_{l,n}] \subseteq [-\alpha a_n, \alpha a_n]$. Then Theorem 3.11 and (3.17) show that $|z| := |z_n|$ is concave on $[x_{l+1,n}, x_{l,n}]$. Hence, the area of the triangle bounded by the x-axis, the vertical line through the point $(x, 0)$, and the line joining the points $(x, |z(x)|)$, $(x_{l,n}, 0)$ is not more than the area bounded by the x-axis, the vertical line through the point $(x, 0)$, and the graph of $|z|$. Thus,

$$\frac{1}{2}|z(x)(x_{l,n}-x)| \leqslant \int_{x}^{x_{l,n}} |z(t)|\, dt. \tag{3.52}$$

We now use the Schwarz inequality, Proposition 3.7, (3.6) (with $P(t) := p_n^2(t)(t-x_{l,n})^{-2}$), and Lemma 3.13 to get

$$\begin{aligned}
&\left\{\frac{1}{2}|z(x)(x_{l,n}-x)|\right\}^2 \\
&\leqslant \left(\int_{x}^{x_{l,n}} \frac{(z(t))^2}{(t-x_{l,n})^2}\, dt\right) \frac{(x_{l,n}-x)^3}{3} \\
&\leqslant c\,\frac{a_n}{n}(x_{l,n}-x)^3 \int_{-\infty}^{\infty} \frac{p_n^2(t)}{(t-x_{l,n})^2}\, w_Q^2(t)\, dt \\
&\leqslant c\,\frac{a_n}{n}(x_{l,n}-x)^3\, \lambda_{ln}\, p_n'^2(x_{l,n}) \\
&\leqslant \frac{c}{a_n}(x_{l,n}-x)^3.
\end{aligned} \tag{3.53}$$

With (3.50), this gives

$$\begin{aligned}
|z(x)|^2 &= |p_n(x)\, w_Q(x)|^2 A_n^{-1}(x) \\
&\leqslant \frac{c}{a_n}|x_{l,n}-x| \leqslant \frac{c}{n}.
\end{aligned} \tag{3.54}$$

Since $A_n(x) \leqslant cn/a_n$, this completes the proof of Theorem 2.2. ∎

Corollary 2.3 is obvious and so is the second inequality in (2.12). Under conditions on Q much less restrictive than ours, Freud proved in [6] (cf. (2.8a)) that

$$\lambda_{kn} \leqslant c\,\frac{a_n}{n}\, w_Q^2(x_{k,n}), \qquad |x_{k,n}| \leqslant c a_n. \tag{3.55}$$

The first inequality in (2.12) follows from this and the identity

$$\lambda_{kn}^{-1} = \rho_n\, p_n'(x_{k,n})\, p_{n-1}(x_{k,n}) = \rho_n A_n(x_{k,n})\, p_{n-1}^2(x_{k,n}) \tag{3.56}$$

together with Proposition 3.7. This completes the proof of Corollary 2.4. ∎

Note added in proof. The author has recently extended the estimate (2.10) for more general Freud-type weights, including $\exp(-|x|^{\alpha})$ when $\alpha > 1$. These results will be published in due course.

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